fessional summer schools in the field; the beginnings of an engineering laboratory; the increased efficiency of military instruction and the summer encampment at Philadelphia in 1876; the erection of a gymnasium, including a lunch room; the admission of women as students.

In 1878, Dr. Runkle resigned the presidency of the Institute and spent the following two years in Europe.

It had been President Runkle's merit to be the first to appreciate the American need of mechanic arts instruction based on principles already successfully applied in Russia. He was primarially interested in it as an invaluable addition to existing engineering courses, but he also saw clearly its great potential significance for general secondary education, and so far as possible, under pressure of other needs, demonstrated this by the inauguration of the School of Mechanic Arts, in which boys of high school age were offered a two years' course, including mathematics, English, French, history, mechanical and free-hand drawing, and shop work. His visit to Europe enabled him to make a study of Continental schools of similar purpose; and the results of this study are embodied in a paper presented to the Society of Arts in April, 1881, on "Technical and Industrial Education Abroad," in an extended contribution to the Report of the Massachusetts Board of Education for 1880-81, and in a "Report on Industrial Education" in 1884. Others have taken a more directly prominent share in the introduction and extension of mechanic arts or manual training in primary and secondary schools, but the actual experiment initiated by him in Boston had in its time wide influence and imitation.

As a teacher of mathematics, Professor Runkle found his highest usefulness and most congenial vocation,—a vocation to be happily continued for not less than twenty-one years. His teaching was characterized by stimulating, luminous, unconventional exposition, by quick, incisive questioning, by warm personal interest in his students, and by a constant substratum of uplifting earnestness and dignity. None of his students could fail to acquire admiring affection; very few could withstand the incentive to work.

Professor Runkle was a man of much intellectual quickness and strength, of ardent, but in later years serene, temperament, of warm and generous affection, of cordial, unaffected courtesy, in all the relations of life a sincere and loyal gentleman. Throughout his early and middle life he was a pioneer, first in the struggle for his own education and that of his brothers, next in the establishment and continuance of a much needed, but, as it turned out, premature mathematical journal, then and for many years in the development of the Massachusetts Institute of Technology, and the introduction of education in the mechanic arts. In all these undertakings his insight and courage were invaluable. He made President Rogers' plans for the Institute his own. He held steadfastly to its fundamental ideals, and, taking account of his scanty resources, made remarkable progress toward their fulfillment. The main changes he initiated have been abundantly justified by time, and he lived to see their fulfillment.

He was elected a Fellow of the Academy on the 26th of May, 1857, and served one year (1877-78) as Councillor.

CONCERNING THE BIBLIOGRAPHY OF MATHEMATICS.*

By PROFESSOR J. W. A. YOUNG.

I. The Character of the Bibliographic Problem in Mathematics.

Mathematics is without doubt the most impersonal of all the branches of human thought. "Views" are not tolerated within its domain; "schools of thought" in the ordinary acceptance of the term are unknown. Conjectures, beliefs, moral certitudes, even, are not given a place among the truths of mathematics, though they are usually more or less distinct stages in the discovery of such truths. Only those statements are admitted to the rank of established mathematical truths, which are accepted by every normal mind to which they are properly presented.

Mathematics has been characterized as the only science which is not concerned about the truth or falsity of its data. The form of reasoning is: *If these premises are true, and if no other premises are taken into account, then such a conclusion follows.* The second *if* is very important and is a part of every mathematical inference.

A consequence of the severe standard thus set is that mathematics revises its results little. The statements of Euclid are as valid today as two thousand years ago The mathematician does not overturn the results of his predecessors; he *extends* them. He may find that the earlier work did not cover all the possibilities that have been seen later, but he does not on that account impugn the validity of what was done, so far as it went.

A good example is found in Euclid's postulate of parallels, viz., "*Given a point and a straight line, there is one (and only one) straight line passing through the point and parallel to the given straight line.*"† For many centuries this postulate was accepted as a logical necessity in Geometry, but during the 19th century‡ it was clearly seen that there was no reason why other assumptions should not be made, either that *no* parallel can be drawn through the given point, or that more than one can be drawn and a consistent Geometry developed. Thus arose the non-Euclidean Geometries. But the validity of the Euclidean Geometry was not at all called into question. The change made was that what was once *the* geometry became *a* geometry among several.

I have cited this as an illustration of the general statement: *The growth of mathematics has been an evolution without a revolution.* Mathematics has as yet had no Copernicus, and expects none.

All this has had a profound influence on mathematical literature. What is written is either a first presentation of a new result, or a representation of it

*An address delivered before the Chicago Bibliographical Society, April 30, 1903.

†By parallel straight lines, we understand here, straight unterminated lines in the same plane, which do not intersect.

‡Simon asserts (*Jahresbericht d. deutschen Math. Ver.* 1890, page 39) that the earliest established recognition that the parallel axiom is not a logical necessity was made by Gauss about 1792, who influenced the Bolyais and Lobatschewsky.

for one of several purposes that need not be enumerated here. The great geniuses of mathematics have not headed opposed and contending schools, but in harmonious, supplementary activity have added new truths to the mathematical treasury. In mathematics, therefore, questions of Bibliography center not about men but about topics. We need not expect to find thousands of works of comment and controversy written in a few years about the theories of any mathematician, however great.

II. What Are the Mathematician's Bibliographic Needs?

It is evident from what has just been said that the mathematical investigator needs primarily *topical* bibliographies. To inform ourselves thoroughly as to what has already been done is an almost indispensable preliminary to any serious research. This is needed both to enable the investigator to avail himself of what has already been accomplished, and also to prevent duplication. In other branches the independent re-doing of nominally the same piece of work is likely to be accomplished in such different form and spirit that the value of the one work is diminished little by the existence of the other. In mathematics real duplication is quite possible and, in fact, has frequently occurred.

III. How the Need Is Met.

1. The first refuge is always the *Fortschritte der Mathematik*, issued annually, giving brief descriptions of all available mathematical publications of the year in question. (I shall refer to this journal hereafter simply as the *Fortschritte*). It covers the period back to 1870, and references to all the earlier publications of importance are to be expected in various connections in the publications of that period. These references often amount to a very fair bibliography of the topic to date. I shall mention other publications (made and projected), covering the literature since 1800, which is by far larger and more important part of the whole.

The chief disadvantage of the *Fortschritte* is that it is about three years behind time. (The volume for 1900, for example, has just been completed.) Several other publication bring us nearer to date.

2. The most important of these is the *International Catalogue of Scientific Literature*, published for the International Council by the Royal Society of London. The first annual issue (for 1901) was published November, 1902. It gives titles only, classified on a decimal system, four digits in each number. The issue contains two catalogues, subject and author, each given in four languages, English, French, German, Italian.*

3. The International Congress for Bibliography of Mathematical Science held in Paris, 1889, appointed a commission to secure a list of titles of works from 1800 to 1889, with decennial supplement thereafter. The work done has taken the form of card titles,—published in series of abont 1000 each,—and the publication is still in progress.

4. *The Revue Semestrielle des Publications Mathématiques*, published under the

*A fuller description of this important catalogue will follow in a later number of the Monthly.

auspices of Mathematical Society of Amsterdam, now in its twelfth volume, comes nearest to covering the literature to date. The part for April to October, 1892, for example, appeared in February, 1903. The journals only are covered, and a few words of abstract are sometimes given.

5. *The Bolletino di Bibliografia e storia delle scienze matematiche* (in its 6th volume) is given mainly to reviews of new books.

These are the chief periodicals given up to bibliographic work exclusively. Among them, the *Fortschritte*, the oldest, still remains the most valuable aid so far as it goes, on account of its abstracts which range from a few lines to several pages according to circumstances.

A comparison of numbers of titles may be of some interest. The Royal Society Catalogue for 1901 contains 1506 titles in pure mathematics (2096 in Astronomy). The initial volume is necessarily somewhat incomplete, as quite a complex organization of national bureaus, readers, etc., has to be got into running order. The *Fortschritte* for 1900 contains (estimate) about 2600 titles, including applied mathematics, geodesy, and astronomy.

6. A most serviceable aid to the working mathematician, especially on the bibliographic side, is *L'Intermédiaire des Mathématiciens* (monthly; founded in 1894). This journal is intended as a medium for interchange of information between working mathematicians. The form is that of questions and answers. The questions represent genuine needs of the writer arising in his researches; often they are directly bibliographical, while others relate to solution of specific problems. The answers are bibliographical when possible; that is, references which will put the questioner on the track of what he wants; though if a reader can give the derived solution himself, but knows of no previous publication of it, the solution itself will be published as reply if brief; if extended, it is published elsewhere, and reference made in *Intermédiare*. The *Intermédiaire* is now completing its first decade; it has met a cordial reception among mathematicians and has done good service in making generally available many pieces of information which would otherwise have remained unutilized.

7. Another phase of general mathematical bibliography is that of classified collections of results, with references to sources for proofs. Many such have been made of varying character and thoroughness, dealing with only a small field; others attempting to cover the entire range of mathematics. I can not now enter upon an enumeration of them; to prepare an exhaustive list would be a bibliographic task in itself. But I mention simply two: Carr, Synopsis of Pure Mathematics (London, 1886, pp. xxxvii, 935), and Hagen, Synopsis der Höheren Mathematik. The latter work by J. G. Hagen, Director of the Observatory, Georgetown College, Washington, D. C., is a stupendous undertaking, planned to be completed in four quarto volumes, of which two have appeared, and form most valuable works of reference.

8. But no one man can prepare the best possible compendium of all mathematics, or even of any large branch of it. Realizing this there is being prepared, under the auspices of the academies of science in Munich and Vienna, and

the Scientific Society of Göttingen, jointly, an *Encyclopedia of Mathematics* which is to be a compendium of the status of mathematical science today, with bibliographic references since the beginning of the nineteenth century. The work is planned in seven volumes, each in many parts, and over sixty prominent mathematicians are connected with the first three volumes, dealing with so-called pure mathematics (78 topics, each entrusted to a specialist). The work is now in process of appearing (first installment issued in 1898) and constitutes the best general survey of the field of mathematics which exists at present.*

9. Parallel with the encyclopedia, which is purely bibliographical, its publishers (Teubner, Leipzig) are also getting out a *series of mathematical text-books*, which will include proofs as well as results, and which are intended to give a fuller survey of the present state of the various topics taken up than can be given within the narrow limits of the encyclopedia. The authors are mathematicians of high rank. In a number of instances the writer of a section in the encyclopedia furnishes a corresponding treatise as text-book, and the series will thus constitute a most valuable supplement to the encyclopedia.*

10. The *Jahresbericht der Deutschen Mathematiker Vereinigung*, founded 1891, publishes excellent, sometimes elaborate reports on the development and on the present status of special topics, which reports often constitute most valuable bibliographies. Thus there have already been published reports on the Theory of Invariants, the Theory of Algebraic Functions, the Theory of Algebraic Number-fields, Synthetic Geometry, text-books on Infinitesimal Calculus, Theory of Probabilities, Point Manifolds, Kinetic Problems of Scientific Technology, Development According to Oscillatory Functions, and a number of others are announced for the near future.†

11. There has just been published (1903) a German work, *Mathematischer Bücherschatz*, by Wölffing (Teubner, Leipzig), which aims to give a complete list of all non-periodical advanced mathematical works of the world, published in the Nineteenth century. The titles are arranged under 313 heads, and alphabetically by authors under these heads.

12. The *Bibliotheca Mathematica* (Stockholm), though primarily devoted to the history of mathematics, occasionally contains good bibliographies of special topics.

IV. Classifications of Mathematics.

Prerequisite to any general bibliographic work like any of those men-

*A fuller account of the encyclopedia and the series will be given in a subsequent number of the MONTHLY.

†Meyer: *Bericht ueber d. gegenwaertigen Stand d. Invarianten-theorie, 1890.*
Brill u. Noether: *Die Entwickelung d. Theorie d. algebraischen Functionen in aelterer u. neuerer Zeit, 1893.*
Hilbert: *Die Theorie d. algebraischen Zahlkoerper, 1895.*
E. Koetter: *Die Entwickelung d. synthetischen Geometrie, 1897.*
Bohlmann: *Uebersicht ueber die wichtigsten Lehrbuecher d. Infinitesimalrechnung von Euler bis auf die heutige Zeit, 1897.*
Czuber: *Die Entwickelung der Wahrscheinlich keitstheorie, 1898.*
Schoenflies: *Entwickelung d. Lehre d. Punktmannigfaltigkeiten, 1899.*
Heun: *Die Kinetische Probleme d. wissenschaftlischen Technik, 1900.*
Burkhardt: *Entwickelungen nach oscillirenden Funktionen, 1901.*

tioned, is some classification of mathematical topics. The *Encyclopedia* entitles its seven volumes as follows (thus forming a first classification):

I. Arithmetic and Algebra II. Analysis III. Geometry	Pure Mathematics.
IV. Mechanics V. Theoretic Physics VI. Geodesy and Mathematical parts of Geophysics and Astronomy VII. Philosophical, Historical, and Pedagogical Questions.	Applied Mathematics.

Each volume is subdivided into twenty or more topics which are assigned to various writers, each one of whom organizes the matter within his own topic as may seem best to him.

A somewhat similar classification is continued in the *Fortschritte*, while those of the *Reportoire* and of the Royal Society Catalogue are each more elaborate. The former has five indices (example, L'3bα1), while the latter is on a decimal system with four digits (example, 3470).

The symplicity of the latter system produces a favorable first impression, but as it has been made public only so recently it is not yet possible to speak concerning its merits in comparison with the earlier schemes of classification, or to prognosticate how well it will be found adapted to the actual work of classification and reference.

V. Needs of Mathematical Bibliography.

The bibliographical helps which are periodical (appearing annually or more frequently) cover the whole field in each issue. Topical bibliographies are also desirable with sub-classifications and reference to minor portions of works on other topics, which deal with the topic in hand.

The *Fortschritte* makes little attempt at sub-classification. The Royal Society Catalogue has a more elaborate classification and some cross references—but in a comprehensive bibliography of all mathematics sub-classification cannot with advantage be carried to the same extent as in the bibliography of a special topic.

Bibliographies of the latter type may achieve two important ends, viz:

1. All the material in a single classification independently of the year of publication.

2. Greater sub-classification than is possible in a more general bibliography.

Topical bibliographies, whether of large subjects or of narrow, highly special fields are now perhaps the most important bibliographic disiderata in mathematics. Much work of this sort has already been done. It would be a bibliographic undertaking of considerable magnitude and importance to prepare a fairly complete list of such special bibliographies.*

*I mention simply a few by American writers:

At the International Congress of Mathematicians held at Paris in 1900, Mr. Ed. Maillet recommended the preparation and publication of bibliographic notices for the assistance of those who wish to take up the study of specific problems. He gives as a model:

"The last theorem of Fermat: $x^m+y^m \neq z^m$, $m>2$. Fermat announced without proof the theorem: The indeterminate equation $x^m+y^m=z^m$ cannot be satisfied by integers if $m>2$. This theorem has not yet been completely solved. To begin its study read:

Serret: *Algébra Supérieure;*
Legendre: *Mém. de l' Institute, 1823;*
Dedekind-Dirichlet: *Zahlentheorie;*
Bachmann: *Kreistheilung;*
(or, in place of the last two books, Bachmann: *Zahlentheorie*);
Kummer: *Jour. d. Math.* t. XVI, *Jour. f. Math.* 1837, 1846, 1847, 1850, *Abh. d. Wiss. z. Berlin*, 1857;
Mirimanoff: *Journal für Mathematik;*
Hilbert, Maillet: *Mém. de l' Assoc. franc. pour l' avanc des Sciences*, 1897; *Comptes rendus de l'Acad. des Sc.*, July, 1899, and *Acta. Mat.*, 1900.

The first attack might be restricted to an attempt to prove that $x^{\lambda t} + y^{\lambda t} = z^{\lambda t}$, λ a prime, is impossible in prime integers, for every t superior to a certain limit function of λ; the theorem is already established when x, y, z are prime to λ, and among themselves.

Sketches like the above for a great number of mathematical problems, whether difficult or not, outlining in a few lines the state of the question, and the problems to be solved, are very desirable; *Intermédiaire* stands ready to publish them."

On the same general line, Hilbert read at the Congress a very important and inspiring paper on the future problems of mathematics. This paper may be found in French, in the *Congress Reports*, 1900; in German, in the *Gött. Nachrichten*, 1900; also in the *Archiv. f. Math. u. Physik*, 1900; in English, in the *Bulletin of the American Mathematical Society*, 1902.

What precedes relates to the newer topics and fields, those which are the scene of present mathematical activity, and where a good bibliography may contribute directly to further investigation.

Bibliographies in the field of the older and settled questions are, however, also desirable, both because these questions stand in more or less close relation to unsettled questions, and also because a careful study of the genesis of any notion, its treatment and development to a completed form is always instructive and stimulating.

Halsted: *Bibliography of Non-Euclidean Geometry*, American Journal of Mathematics, 1879.
Miller: *On Recent Progress in the Theory of Groups of a Finite Order*, Bulletin American Mathematical Society, 1899; *of Infinite Order, ibid.*, 1900.
Dickson: *Report on the Recent Progress in the Theory of Linear Groups*, Bulletin American Mathematical Society, 1899.
The Known Systems of Simple Groups and their Isomorphisms, Report of the Paris Congress, 1900, page 225.
Easton: *Bibliograyhy of Substitution Groups*, published separately, Philadelphia, 1902.
Macfarlane: *Bibliography of Quaternions* (in preparation).

A DISCUSSION OF THE CASES WHEN TWO QUADRATIC EQUATIONS INVOLVING TWO VARIABLES CAN BE SOLVED BY THE METHOD OF QUADRATICS.*

By MISS ADELAIDE DENIS, Graduate Student, Colorado College.

1. The treatment of simultaneous quadratics in our elementary text-books is most unsatisfactory to the teacher. The author sometimes begins with the theorem, "The solution of a system of quadratic equations involving two variables in general requires the solution of a biquadratic." More often, no mention is made of the general theorem. Three cases are stated where special devices make possible the solution by quadratics. Then follows a set of problems, some under these three heads, many not. The pupil is left to use his ingenuity, with more or less suggestion from others, in solving them.

Chrystal says: "A moderate amount of practice in solving puzzles of this description is useful as a means of cultivating manipulative skill, but he (the student) should beware of wasting his time over what is, after all, merely a chapter of accidents."

The purpose of the following paper is to attempt to remove the problem from the category of a "chapter of accidents," or the realm of puzzles, and make it possible for the teacher, if not the pupil, to determine when the solution of such a system is possible by the method of quadratics. The conditions that the given equations can be solved entirely by the method of quadratics are obtained. In certain cases where these tests fail, it is still possible to obtainq uadratic factors, or a linear factor which will give a partial solution. The latter case is of especial interest, as indicated by the discussions in Vol. VI, pages 13-14, and Vol. VII, page 169, of THE AMERICAN MATHEMATICAL MONTHLY.

For the following work, Chrystal's *Text-book of Algebra*, Third Edition, Part I, pages 416-417; Burnside and Panton's *Theory of Equations*, Third Edition, pages 129-130; and Dr. K. L. Bauer's article in *Hoffmann's Zeitschrift*, 1874, page 317, have been found most helpful.

2. Let the given system of equations be represented by

$$ax^2+by^2+cxy+dx+ey+f=0.....(\text{I})$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0.....(\text{II}).$$

Following Chrystal's suggestion as to method of solution, let

$$cy+d=p, \quad by^2+ey+f=q,$$

$$c'y+d'=p', \quad b'y^2+e'y+f'=q'.$$

Substituting these values in (I) and (II), we get

* Submitted through Professor Cajori and somewhat condensed by Editor Dickson.

$$ax^2+px+q=0, \quad a'x^2+p'x+q'=0.$$

Eliminating x^2, and finally, also x itself, we get

$$(aq'-a'q)^2-(ap'-a'p)(pq'-p'q)=0.$$

Substituting the values assumed for p and q,

$$[a(b'y^2+e'y+f')-a'(by^2+ey+f)]^2-\{[a(cy'+d')$$
$$-a'(cy+d)][(cy+d)(b'y^2+e'y+f')-(c'y+d')(by^2+ey+f)]\}=0 \text{....(III)}.$$

In expanding the last form the coefficients are such that determinants can be used to advantage. The determinant $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ or $ab'-a'b$ will for convenience be written in the form $[ab']$. The equation takes the form

$$Ay^4+By^3+Cy^2+Dy+E=0\text{.....}(Q).$$

where

$$A=[ab']^2+[ac'][bc'],$$

$$B=2[ae'][ab']-[ac'][ce']+[ac'][bd']+[ad'][bc'],$$

$$C=[ae']^2+2[ab'][af']-[ac'][cf']-[ac'][de']-[ad'][ce']+[ad'][bd'],$$

$$D=2[ae'][af']-[ac'][df']-[ad'][cf']-[ad'][de'],$$

$$E=[af']^2-[ad'][df'].$$

3. If the quartic (Q) is separable into quadratic factors, the quartic may be said to be irreducible or reducible, according as the coefficients do or do not involve irrational numbers.

I. Irreducible quartic:

(1) $[(r+\sqrt{m})y^2+ny+p][(r-\sqrt{m})y^2+ny+p]=0.$

(2) $[my^2+(r+\sqrt{n})y+p][my^2+(r-\sqrt{n})y+p]=0.$

(3) $[my^2+ny+(r+\sqrt{p})][my^2+ny+(r-\sqrt{p})]=0.$

II. Reducible quartic:

(1) Solvable by method of quadratics. The same forms as I, if $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$ are rational.

(2) Solvable by factoring of quartic, but not by the method of quadratics, pure and simple:

(a) $(my^2+ny+p)(m'y^2+n'y+p')=0.$

(b) $(my^3+ny+py+q)(m'y+q')=0.$

4. All equations of the first form I, (1), can be written

$$(ry^2+ny+p)^2-my^4=0, \text{ or } (Xy^2+\frac{D'}{2}y+1)^2-Yy^4=0\ldots\ldots(\text{IV}),$$

in which the absolute term is obtained by dividing both factors by p; $D'=D/E$, D and E as in (Q), X and Y to be found.

Expanding (IV) and comparing the coefficients with the coefficients of the same powers in (Q), after dividing (Q) by E,

$$A/E=X^2-Y, \quad D'X=B/E, \quad D'/4+2X=C/E.$$

Equating the values of X derived from the last two,

$$D^3+8BE^2=4CDE\ldots\ldots(J).$$

Therefore (J) is the condition that the quartic be capable of being resolved into factors of the first form.

5. All equations of the second form I, (2), can be written

$$(my^2+ry+p)^2+ny^2=0 \text{ or } [\sqrt{(a)}y^2+Xy+\sqrt{E})-Yy^2=0\ldots\ldots(\text{V}),$$

in which A and E are the same as in (Q), X and Y to be found. Expanding and comparing the coefficients with the coefficients of the same powers in (Q),

$$2\sqrt{a}\,X=B, \quad X^2+2\sqrt{(AE)}-Y=C, \quad 2X\sqrt{E}=D.$$

$$\therefore B^2E=D^2A\ldots\ldots(K).$$

Therefore (K) is the condition that the quartic (Q) shall be capable of being resolved into factors of the second form.

6. All equations of the third form can be written

$$(my^2+ny+r)^2-p^2=0 \text{ or } (y^2+\frac{B'}{2}y+X)^2-Y=0\ldots\ldots(\text{VI}),$$

in which $B'=B/A$, B and A as in (Q), X and Y to be found. Expanding and comparing coefficients with the coefficients of the like powers in (Q), after dividing by A,

$$2X+B'^2/4=C/A, \quad B'X=D/A, \quad X^2-Y=E/A.$$

$$\therefore B^3+8A^2D=4ABC\ldots\ldots(L).$$

Therefore (L) is the condition that the quartic (Q) be capable of being resolved into factors of the third form.

The test (L) is the one obtained by Dr. K. L. Bauer in his article referred to in the introduction.

As an example under the case I of irreducible quartic, take

$$5y^2+2y-x-3=0, \quad x^2+6x+40y^2+16y+20=0.$$

Then (Q) becomes

$$25y^4+20y^3+44y^2+16y+11=0.$$

Trying successively the tests (J), (K), (L), the last one gives the identity

$$(20)^3+8.(25)^2.16=4.25.20.44.$$

Then $X=D/B=\frac{4}{5}$, $Y=X^2-E/A=\frac{1}{5}$.

$$[y^2+\tfrac{2}{5}y+\tfrac{4}{5}+\sqrt{\tfrac{1}{5}}][y^2+\tfrac{2}{5}y+\tfrac{4}{5}-\sqrt{\tfrac{1}{5}}]=0.$$

7. If (Q) is separable into rational factors, these factors may be both quadratic, or one linear and one cubic. If the factors are quadratic, the roots of the quartic may be all irrational, two irrational and two rational, or all rational. The general form for such a quartic is,

$$(my^2+ny+p)(m'y^2+n'y+p')=0.....(\text{VII}).$$

If the radical terms disappear in the three forms already considered, the resulting forms are special cases under (VII). If, therefore, the tests (J), (K), or (L) give a result in which $\sqrt{Y}$ is rational, the roots of the quartic may be obtained by the method used for the irreducible case.

8. Consider next the special devices that lead to the factoring of quartic, when the previous direct tests fail and the quartic cannot be solved by the method of quadratics.

If the quartic (Q) is separable into two quadratic factors, the general form (VII) has been given. One of these factors may be obtained as follows. Dividing (Q) by A, the coefficient of y^4, and transforming the resulting equation so as to remove fractional coefficients, if this be necessary, let the resulting quartic be

$$y^4+a_1y^3+a_2y^2+a_3y+a_4=0.....(Q').$$

If (Q') is separable into quadratic factors, let

$$y^2+ay+\beta.....(P)$$

be one of those factors. Dividing the first member of (Q') by this factor, the remainder is

$$[(a_3-a_2\beta+a\beta)-(a_2a-a\beta-a_1a^2+a^3)]y+a_4-(a_2\beta-\beta^2-a_1a\beta+a^2\beta)\ldots\ldots(R),$$

and the quotient is,

$$y^2+(a_1-a)y+(a_2-\beta-a_1a+a^2)\ldots\ldots(P').$$

If $(R)=0$, then (P) and (P') are the quadratic factors of (Q').

The two conditions that $(R)=0$, identically, are

$$(1)\quad a_3-a_1\beta+a\beta=a_2a-a_1a^2+a^3-a\beta,$$

$$(2)\quad a_4=a_2\beta-\beta^2-a_1a\beta+a^2\beta.$$

From them

$$\frac{a_3-a_1\beta+2a\beta}{a}=a_2-a_1a+a_2,\quad \frac{a_4}{\beta}+\beta=a_2-a_1a+a_2.$$

Equating the first members, and solving for a,

$$a=(a_3\beta-a_1\beta^2)/(a_4-\beta^2).$$

β must be an integral factor of a_4. Therefore if a value for β can be found by factoring a_4, which will give the above expression for a integral, the factor (P) can be determined and second factor (P') readily found.

If the quartic is reducible, and one factor linear and one an irreducible cubic, the solution cannot be effected by the method of quadratics.

If however, (Q) be divided by A, and the resulting equation transformed to remove fractional coefficients, if necessary, the single real and rational root may be found by the Remainder Theorem. Thus one set of values for x and y in the original equations may be found.

There are some special cases under this general one that are worthy of consideration. Systems of quadratics of the form

$$x^2+y=a,\quad y^2+x=b,$$

have been of especial interest. (See end of §1).

It can readily be shown, by applying the tests (J), (K), and (L), that the resulting quartic does not come under any of the cases solvable directly by the method of quadratics. But sometimes by a special device we can find one set of values for x and y. This is the case when the quartic can be separated into one linear and one irreducible cubic factor. For instance, let the general form of such a system be

$$\begin{aligned}(ax+by+c)^2-d^2&=e-(fx+gy+h)\\(fx+gy+h)^2-e^2&=d-(ax+by+c)\end{aligned}\ldots\ldots(F).$$

That

$$ax+by+c=d,\quad fx+gy+h=e,$$

can be seen by inpection. If the given equations can be arranged as in (*F*), the solution as far as obtainable by this device, is readily completed.

To separate (I) and (II) as given in (*F*), extract the square roots of the first members of (I) and (II) as far as possible. If the remainder obtained in the first is the root obtained in the second, and the remainder in the second the root in the first, the equations may be written in the form (*F*).

A special case of the above general form is

$$x^2-d^2=e-y, \quad y^2-e^2=d-x,$$

in which $x=d$, $y=e$. See the references at end of §1.

Another of these special cases is the one in which the single real and rational root is a quartic can be obtained by the method next explained. The following two theorems will be used, taken from Eugen Netto's *Vorlesungen über Algebra*, Vol. I, page 56:

(1) If all the coefficients, c_1, c_2.....c_n of the polynomial

$$f(y)=y^n+c_1y^{n-1}+\text{....}+c_{n-1}y+c_n$$

are integral and divisible, without a remainder, by a prime number p, but c_n is not divisible by a higher power of p than the first, then $f(y)$ is irreducible.

(2) If all the coefficients, c_1, c_2.....c_n of the polynomial

$$f(y)=y^n+c_1y^{n-1}+\text{.....}+c_{n-1}y+c_n$$

are divisible by a prime number p, but c_{n-1} is not divisible by a higher power of p than the first, $f(y)$ is either irreducible or is reducible to a factor of the first degree and an irreducible factor of the $(n-1)$th degree.

If the coefficients a_1, a_2, a_3, a_4 of the quartic (Q') are divisible by a prime number p, but a_3 is not divisible by a higher power of p than the first, and the quartic is reducible to a linear factor and an irreducible quartic, one root of the quartic, and one set of values of x and y can be ascertained without recourse to the regular algebraic solution. Let the cubic be

$$y^3+c_1y^2+c_2y+c_3=0, \quad c_1=p^nr, \quad c_2=p^mr, \quad c_3=pr_2.$$

If the factor $y+a$ is introduced,

$$y^4+(a+p^nr)y^3+(ap^nr+p^mr_1)y^2+(ap^mr_1+pr_2)y+apr_2=0\text{.....}(M).$$

Let $a+p^nr=B$, $ap^n+p^mr_1=C$, $ap^mr_1+pr_2=D$, $apr_2=E$.

Then a system of quadratic equations giving the quartic (M) is

$$2y^2+By=x, \quad x^2+(4C-B^2)y^2+4Dy+4E=0.$$

9. There are certain forms of simultaneous quadratics which are readily recognized as solvable by the method of quadratics. It may be interesting to place these according to the preceding discussion. The classification given by Chrystal, Algebra, Part I, page 417, has been followed quite closely.

I. Two roots zero. The quartic (Q) assumes the form

$$Ay^4+By^3+Cy^2=0 \text{ or } (Ay^2+By+C)y^2=0.$$

It is evident that the conditions $D=0$, $E=0$ are satisfied if

$$ad'=a'd,\ af'=a'f,\ df'=d'f, \text{ or } a/a'=d/d'=f/f' \ldots\ldots(N).$$

But if $D=0$, $E=0$ tests (J) and (K) are satisfied. Therefore the system of quadratics is solvable by the method of quadratics.

II. Two roots infinity. The quartic (Q) assumes the form

$$Cy^2+Dy+E=0.$$

It is evident that a sufficient condition for the vanishing of A and B is

$$ab'=a'b,\quad ac'=a'c,\quad bc'=b'c, \text{ or } a/a'=b/b'=c/c'.$$

But if $A=0$, $B=0$, the tests (K) and (L) are satisfied; therefore the quadratics are solvable as before.

III. The quartic contains only the even powers of y. The quartic (Q) assumes the form $ay^4+cy^2+E=0$. A sufficient condition for the vanishing of B and D is

$$d=0,\quad d'=0,\quad e=0,\quad e'=0.$$

But these are the conditions that (I) and (II) are homogeneous, and the tests (J), (K), and (L) are satisfied, so that two homogeneous equations and any other systems producing a quartic (Q) containing only the even powers of y, are solvable by the method of quadratics.

IV. If the quartic is a reciprocal equation of the form

$$(1)\quad Ay^4+By^3+Cy^2+By+A=0,$$

then $A=E$, $B=D$. But if $A=E$,

$$\{ab'\}^2+\{ac'\}\{bc'\}=[\{af'\}^2-\{ad'\}\{df'\}],$$

a condition which is satisfied if $b=f$, $b'=f'$, $c=d$, $c'=d'$. If these conditions hold, $B=D$, so that the quartic is reciprocal.

But if $A=E$, $B=D$, the test (K) is satisfied. Therefore a system of quadratics producing a reciprocal quartic of the form (1) is solvable as before.

If the quartic assumes the reciprocal form

$$(2)\quad Ay^4+By^3-By-A=0,$$

the above test (K) does not hold. But if the equation be divided by A, it is seen as in §8 that $a=0$, so that y^2-1 is a factor of the reciprocal equation. The remaining factor is readily found.

V. If the quadratic equations are symmetrical,

$$a=b,\quad d=e,\quad a'=b',\quad d'=e'.$$

Equations (I) and (II) then become, after division by a and a',

$$x^2+y^2+c_2xy+d_1x+d_1y+f_1=0\text{.....(I')},$$

$$x^2+y^2+c_1xy+d_2x+d_2y+f_2=0\text{.....(II')}.$$

Subtracting, $\quad c_3xy+d_3x+d_3y+f_3=0\text{.....(III')}.$

Substituting $x=\mu+\nu$, $y=\mu-\nu$ in (I') and (III'),

$$a_1\mu^2+b_1\nu^2+2d_1\mu+f_1=0\text{.....(IV')},$$

$$c_3\mu^2-c_3\nu^2+2d_3\mu+f_3=0\text{.....(V')}.$$

From (IV') and (V'), ν^2 may be eliminated, and the resulting quadratic in μ readily solved. Or the values for a, a', etc., may be substituted in (III). In this case the terms in ν^3 and ν will vanish and the resulting quartic in ν comes under the special case III just discussed.

AN EXTENSION TO CENTRAL CONICOIDS OF A THEOREM CONCERNING THE SEGMENT OF A SPHERE.

By G. W. GREENWOOD, McKendree College.

Consider the sphere, the cylinder, and the cone whose equations are, respectively,

$$x^2+y^2+z^2=1\text{.....(1)},$$

$$x^2+y^2=1\text{.....(2)},$$

$$x^2+y^2=z^2\text{.....(3)}.$$

It is shown in text books that the volume of (1) included between planes parallel to the plane of x, y is equal to the volume of the segment of (2) diminished by the that of the segment of (3) included between the same planes.

If we employ the equations

$$x^2/a^2+y^2/b^2+z^2/c^2=1.....(4),$$

$$x^2/a^2+y^2/b^2=1.....(5),$$

$$x^2/a^2+y^2/b^2=z^2/c^2.....(6),$$

for (1), (2), (3), respectively, the theorem is still true. If for (4) we substitute either of the equations

$$x^2/a^2+y^2/b^2=1+z^2/c^2.....(7),$$

$$x^2/a^2+y^2/b^2=z^2/c^2-1.....(8),$$

we get the theorems that the volume of a segment of (7) made by planes parallel to that of x, y is equal to the sum of the volumes of the corresponding segments of (5) and (6); and that of a segment of (7) is the volume of the corresponding segment of (6) diminished by that of the segment of (5).

The volume of a segment of (1) made by planes parallel to that of x, y is equal to the sum of the volumes of two cylinders whose bases are the bases of the segment and altitudes half that of the segment, together with the volume of a sphere to which the bases of the segment are tangent.

The corresponding theorems are as follows: The volume of a segment of (4) made by planes parallel to that of x, y is equal to the sum of the volumes of two cylinders whose bases are the bases of the segment and whose altitudes are half that of the segment, together with the volume of an ellipsoid to which the bases of the segment are tangent, similar to (4) and similarly placed.

Furthermore, the volume of a segment of (7) or (8) made by planes parallel to that of x, y is equal to the volumes of two cylinders whose bases are the bases of the segment, diminished by the volume of an ellipsoid to which the bases are tangent, whose axes are proportional to those of (7) or (8) and respectively parallel to them.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

178. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A_n being the arithmetic mean of the nth powers of the numbers less than p and prime to it, find a relation between A_3, A_2 and p.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The number n of positive integers less than p and prime to it is

$$n=p(1-1/a)(1-1/b)(1-1/c)(1-1/d)\ldots\ldots$$

The sum of the squares of all such numbers is

$$S_2=\tfrac{1}{3}p^3(1-1/a)(1-1/b)(1-1/c)(1-1/d)\ldots\ldots+\tfrac{1}{6}p(1-a)(1-b)(1-c)(1-d)\ldots\ldots$$

and the sum of the cubes is

$$S_3=\tfrac{1}{4}p^4(1-1/a)(1-1/b)(1-1/c)(1-1/d)\ldots\ldots+\tfrac{1}{4}p^2(1-a)(1-b)(1-c)(1-d)\ldots\ldots$$

$$\therefore A_2=S_2/n=\tfrac{1}{3}p^2+\tfrac{1}{6}\cdot\frac{(1-a)(1-b)(1-c)(1-d)\ldots\ldots}{(1-1/a)(1-1/b)(1-1/c)(1-1/d)\ldots\ldots}.$$

$$=\tfrac{1}{3}p^2+\tfrac{1}{6}B, \text{ suppose.}$$

$$A_3=S_3/n=\tfrac{1}{4}p^3+\tfrac{1}{4}pB.$$

$$\therefore B=6A_2-2p^2=\frac{4A_3-p^3}{p}.$$

$$\therefore 6A_2p-2p^3=4A_3-p^3, \text{ or } 6A_2p-4A_3=p^3, \text{ or } p^3-6A_2p+4A_3=0.$$

179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left(\frac{q^2}{p}-\frac{p^3}{5q}\right)=0.$$

No solution of this problem has been received.

180. Proposed by the late JOSIAH H. DRUMMOND.

If r/s is such a value of p as makes $m/(p^2-2)$ integral, prove that $(3r+4s)/(2r+3s)$ is another such value, so that an indefinite number of integral values may be obtained.

Also, if r/s is such a value of p as makes $2m/(p^2-2)$ integral, prove that $2(r+s)/(r+2s)$ is also such a value.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

When $p=r/s$, $m/(p^2-2)=ms^2/(r^2-2s^2)$.....(1).

When $p=(3r+4s)/(2r+3s)$, $m/(p^2-2)=m(2r+3s)^2/(r^2-2s^2)$.....(2).

Since (1) is integral, (2) is also, for we can take $m=n(r^2+2s^2)$.

When $p=r/s$, $2m/(p^2-2)=2ms^2/(r^2-2s^2)$.....(3).

When $p=2(r+s)/(r+2s)$, $2m/(p^2-2)=m(r+2s)^2(r^2-2s^2)$.....(4).

Since (3) is integral, (4) is also.

Also solved by the *PROPOSER*.

GEOMETRY.

200. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of eight points of contact of the four common tangents of two concentric coaxial ellipses.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/a^2+y^2/b^2=1$, and $x^2/c^2+y^2/d^2=1$, be the equations to the ellipses, $c>a$, $b>d$.

Since $xx_1/a^2+yy_1/b^2=1$, and $xx_2/c^2+yy_2/d^2=1$, are two equations for the same line, we get $x_2=c^2x_1/a^2$, $y_2=d^2y_1/b^2$.

$\therefore x_1^2/a^2+y_1^2/b^2=1$, and $c^2x_1^2/a^4+d^2y_1^2/b^4=1$, give

$$x_1=\frac{a^2\sqrt{[b^2-d^2]}}{\sqrt{[b^2c^2-a^2d^2]}}=a^2m \text{ (suppose)}, \quad y_1=\frac{b^2\sqrt{[c^2-a^2]}}{\sqrt{[b^2c^2-a^2d^2]}}=b^2n,$$

$$x_2=c^2m, \quad y_2=d^2n.$$

Hence $mx\pm ny\pm 1=0$ represents the four common tangents. While $(c^2m,\ d^2n)$; $(a^2m,\ b^2n)$; $(-a^2m,\ b^2n)$; $(-c^2m,\ d^2n)$; $(-c^2m,\ -d^2n)$; $(a^2m,\ -b^2n)$; $(c^2m,\ -d^2n)$ are the eight points of contact. These eight points are situated on an ellipse, which is the locus required.

Let $x^2/A^2+y^2/B^2=1$ be this ellipse. Then $c^2m/A^2+d^4n^2/B^2=1$, also $a^4m^2/A^2+b^4n^2/B^2=1$.

$$\therefore A^2=\frac{(b^4c^4-a^4d^4)m^2}{b^4-d^4}=\frac{b^2c^2+a^2d^2}{b^2+d^2}, \quad B^2=\frac{(b^4c^4-a^4d^4)n^2}{c^4-a^4}=\frac{b^2c^2+a^2d^2}{a^2+c^2}.$$

$\therefore (b^2+d^2)x^2+(a^2+c^2)y^2=b^2c^2+a^2d^2$ is the locus.

201. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Two plane sections of a right circular cone have their major axes AA', aa' coplanar, and Aa on one generator equal to $A'a'$ on the other. The projections of the sections on any plane perpendicular to the axis are confocal.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $FG=ED=r$, $FD=2A$, $GE=2a$, $\angle EGK=\varphi$, $\angle DFH=\theta$, $\angle GLF=x$, $\angle CGK=\angle CFH=\delta$, $CN=c$, $CM=d$.

Then $GF : LF=\sin x : \sin LGF$, or $r : LF=\sin x : \sin(\delta-\varphi)$.

Also $ED : LD=\sin x : \sin LED$, or $r : 2A+LF=\sin x : \sin(\delta+\varphi)$.

$\therefore LF\sin x=r\sin(\delta-\varphi)=r\sin(\delta+\varphi)-2A\sin x$.

$\therefore A\sin x=r\cos\delta\sin\varphi$.

Similarly, $a\sin x=r\cos\delta\sin\theta$. $\therefore A\sin\theta=a\sin\varphi$.

$GN+EN=2a=2c\cos\delta\sin\delta\cos\varphi/(\cos^2\varphi-\cos^2\delta)$.

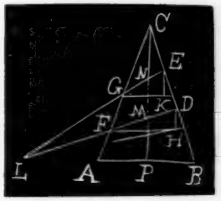
$\therefore c=a(\cos^2\varphi-\cos^2\delta)/\cos\delta\sin\delta\cos\varphi$,

$d=A(\cos^2\theta-\cos^2\delta)/\cos\delta\sin\delta\cos\theta$.

The axes of the projections on any plane perpendicular to CP are $A\cos\theta$ and B, and $a\cos\varphi$ and b, respectively.

$$\left.\begin{aligned} &x^2\cos^2\varphi(\tan^2\delta-\tan^2\varphi)+y^2\tan^2\delta+2cx\sin\varphi=c^2 \\ &x^2\cos^2\theta(\tan^2\delta-\tan^2\theta)+y^2\tan^2\delta+2dx\sin\theta=d^2 \end{aligned}\right\}$$ are the equations to the ellipses.

$$\therefore A^2=\frac{d^2\cos^2\theta\sin^2\delta\cos^2\delta}{(\cos^2\theta-\cos^2\delta)^2},\quad B^2=\frac{d^2\cos^2\theta\cos^2\delta}{\cos^2\theta-\cos^2\delta}.$$



$$a^2=\frac{c^2\cos^2\varphi\sin^2\delta\cos^2\delta}{(\cos^2\varphi-\cos^2\delta)^2}=b^2=\frac{c^2\cos^2\varphi\cos^2\delta}{\cos^2\varphi-\cos^2\delta}.$$

Substituting values of d and c,

$$A^2\cos^2\theta-B^2=\frac{d^2\cos^2\theta\sin^2\theta\cos^4\delta}{(\cos^2\theta-\cos^2\delta)^2}=A^2\sin^2\theta\cot^2\delta.$$

$$a^2\cos^2\varphi-b^2=\frac{c^2\cos^2\varphi\sin^2\varphi\cos^4\delta}{(\cos^2\varphi-\cos^2\delta)^2}=a^2\sin^2\varphi\cot^2\delta=A^2\sin^2\theta\cot^2\delta.$$

$\therefore A^2\cos^2\theta-B^2=a^2\cos^2\varphi-b^2$, and the sections are confocal.

202. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The equations $\sqrt{[l\alpha]}+\sqrt{[m\beta]}+\sqrt{[n\gamma]}=0$ and $l\beta\gamma+m\alpha\gamma+n\beta\alpha=0$ represent ellipses. If a, b, c are the sides of the triangle of reference, transform to Cartesian coördinates and find area of each ellipse.

Solution by the PROPOSER.

$\sqrt{[l\alpha]}+\sqrt{[m\beta]}+\sqrt{[n\gamma]}=0$, or

$$l^2a^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\alpha\gamma-2lm\alpha\beta=0\ldots\ldots(1),$$

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=0\ldots\ldots(2).$$

Let ABC be the triangle of reference, B the origin, BC, BA the axes of coördinates. Then $\alpha = y\sin B$, $\gamma = x\sin B$, also $a\alpha + b\beta + c\gamma = ac\sin B$.

$\therefore \beta = [ac\sin B - a\alpha - c\gamma]b = \sin B[ac - ay - cx]/b$.

These values of α, β, γ in (1) and (2) give

$$[bn+cm]^2x^2+[am+bl]^2y^2+2[acm^2+abmn+bclm-b^2nl]xy$$

$$-2[ac^2m^2+abcmn]x-2[a^2cm^2-abclm]y+a^2c^2m^2=0 \ldots\ldots(3).$$

$$clx^2+any^2+[al+cn-bm]xy-aclx-acny=0 \ldots\ldots(4).$$

(3) reduces to $\{[bn+cm]x+[am+bl]y-acm\}^2-4b^2nlxy=0$.

If $A'x^2+B'y^2+2H'xy+2G'x+2F'y+C'=0$ be the equation (general) to an ellipse and $\triangle$ its discriminant, then area $= \dfrac{\pi\triangle\sin B}{[A'B'-H'^2]^{\frac{3}{2}}}$ where the axes are inclined at an angle B.

For (3), $\triangle = -4a^2b^4c^2l^2m^2n^2$, $A'B'-H'^2 = 4b^2lm[abn+acm+bcl]$.

$$\therefore \text{Area} = \tfrac{1}{2}\pi a^2bc^2\sin B\sqrt{\frac{lmn}{[abn+acm+bcl]^3}}.$$

If $l=\cos^2\tfrac{1}{2}A$, $m=\cos^2\tfrac{1}{2}B$, $n=\cos^2\tfrac{1}{2}C$, the ellipse becomes the in-circle.

$$\text{Area} = \frac{\pi a^2bc^2\cos\tfrac{1}{2}A\,\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\sin B}{2\sqrt{\{[bc\cos^2\tfrac{1}{2}A+ac\cos^2\tfrac{1}{2}B+ab\cos^2\tfrac{1}{2}C]^3\}}}$$

$$= \frac{\pi acs\sqrt{\{s[s-a][s-b][s-c]\}}\sin B}{2\sqrt{(\{[s-a]+s[s-b]+s[s-c]\}^3)}}, \text{ where } s=\tfrac{1}{2}[a+b+c],$$

$$= \tfrac{1}{2}\pi ac\sin B\sqrt{\{s[s-a][s-b][s-c]\}}/s^2$$

$$= \pi\left(\frac{\text{area of triangle}}{s}\right)^2 = \pi r^2.$$

For (4), $\triangle = -\tfrac{1}{2}a^2bc^2lmn$, $A'B'-H'^2 = \tfrac{1}{4}\{4acln-[al+cn-bm]^2\}$.

$$\text{Area} = \frac{2\pi a^2bc^2lmn\sin B}{\sqrt{([\{4acln-[al+cn-bm]^2\}^3)}}.$$

Let $l=\sin A$, $m=\sin B$, $n=\sin C$, and the ellipse becomes the circum-circle.

$$\text{Area} = \frac{2\pi a^2bc^2\sin A\,\sin^2B\sin C}{\{4ac\sin A\sin C-[a\sin A+c\sin C-b\sin B]\}^{\frac{3}{2}}}$$

$$=\frac{2\pi a^2b^3c^3\sin A}{\sqrt{(\{4a^2c^2-[a^2+c^2-b^2]^2\}^3)}}=\frac{\pi a^2b^3c^3\sin A}{32\sqrt{(\{s[s-a][s-b][s-c]\}^3)}}$$

$$=\frac{\pi a^2b^2c^2}{[4\text{ area of triangle}]^2}=\pi R^2.$$

CALCULUS.

160. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly towards the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

Note by J. E. SANDERS, Hackney, Ohio.

The latter part of the solution given in the April MONTHLY is not correct. The dog cannot run towards the fox at all times and keep between the fox and the vertex of the hill. If we suppose only that he runs so that he is directly between the fox and the vertex, the following is the solution:

Let the surface of the hill be rolled out flat. Then the radius of the fox's path$=100\sqrt{2}$ feet$=a$. Put $n=\frac{5}{6}$, $s=$length of dog's path.

$$ds=[a/n]d\theta=\sqrt{\{r^2+[dr/d\theta]^2\}}.$$

$\therefore d\theta=\dfrac{ndr}{\sqrt{[a^2-n^2r^2]}}$, which gives $\theta=\sin^{-1}[nr/a]$ when integrated. When the dog catches the fox $r=a$, and

$\therefore \theta=\sin^{-1}n$ and $s=[a/n]\theta=120\sqrt{2}\sin^{-1}[\frac{5}{6}]=167.178$ feet.

No solution of Problem 165 has been received. See April number, 1903, page 115.

166. Proposed by T. N. HAUN, Mohawk, Tenn.

Find the volume of the solid formed by the revolution of the curve $[y^2+x^2]=a^2[x^2-y^2]$ round the axis of x.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$[y^2+x^2]^2=a^2[x^2-y^2].$$

$$\therefore y^4+2x^2y^2+x^4=a^2x^2-a^2y^2. \quad \therefore y^2=\tfrac{1}{2}\{a\sqrt{[a^2+8x^2]}-a^2-2x^2\}.$$

$$\text{Volume}=2\int_0^a \pi y^2 dx=\pi\int_0^a \{a\sqrt{[a^2+8x^2]}-a^2-2x^2\}dx$$

$$=\pi\left[\frac{ax}{2}\sqrt{[a^2+8x^2]}+\frac{a^3}{4\sqrt{2}}\log\{\sqrt{[a^2+8x^2]}+2\sqrt{2}\,x\}-a^2x-\tfrac{2}{3}x^3\right]_0^a$$

$$=\frac{\pi a^3}{2}\left[\frac{1}{2\sqrt{2}}\log[3+\tfrac{1}{2}\sqrt{2}]-\tfrac{1}{3}\right]=\frac{\pi a^3}{2}\left[\frac{1}{2\sqrt{2}}\log[1+\sqrt{2}]^2-\tfrac{1}{3}\right]$$

$$=\frac{\pi a^3}{2}\left[\frac{1}{\sqrt{2}}\log[1+\sqrt{2}]-\tfrac{1}{3}\right].$$

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Integrate, $\int_0^a\int_0^b\int_0^c \frac{zdxdydz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$.

Solution by the PROPOSER.

$$\int_0^a\int_0^b\int_0^c \frac{zdxdydz}{\sqrt{\{(x^2+y^2+z^2)^5\}}}=\tfrac{1}{3}\int_0^a\int_0^b\left[\frac{1}{\sqrt{\{(x^2+y^2)^3\}}}\right.$$

$$\left.-\frac{1}{\sqrt{\{(c^2+x^2+y^2)^3\}}}\right]dxdy$$

$$=\frac{b}{3}\int_0^a\left[\frac{1}{x^2\sqrt{(b^2+x^2)}}-\frac{1}{(c^2+x^2)\sqrt{(b^2+c^2+x^2)}}\right]dx$$

$$=-\tfrac{1}{3}\left[\frac{\sqrt{(b^2+x^2)}}{bx}+\frac{1}{c}\tan^{-1}\left(\frac{bx}{c\sqrt{(b^2+c^2+x^2)}}\right)\right]_0^a$$

$$=-\tfrac{1}{3}\left[\frac{\sqrt{(a^2+b^2)}}{ab}+\frac{1}{c}\tan^{-1}\left(\frac{ab}{c\sqrt{(a^2+b^2+c^2)}}\right)\right]+\infty=\infty.$$

DIOPHANTINE ANALYSIS.

113. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the four least integral numbers such that the difference of every two of them shall be a square number.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and J. A. SANDERS, Hackney, Ohio.

Let a, m^2+a, n^2+a, p^2+a be the numbers.

Then $p^2+a-a=p^2$, $n^2+a-a=n^2$, $m^2+a-a=m^2$, $p^2+a-n^2-a=p^2-n^2$, $p^2+a-m^2-a=p^2-m^2$, $n^2+a-m^2-a=n^2-m^2$.

Three of the conditions are satisfied and we must make p^2-n^2, p^2-m^2, n^2-m^2 all squares. This is done in Vol. IX, No. 4, pages 113-114.

$\therefore p=697d$, $n=185d$, $m=153d$.

$\therefore$ The numbers are a, $a+23409d^2$, $a+34225d^2$, $a+485809d^2$.

Let $a=d=1$, and the numbers are 1, 23410, 34226, 485810.

114. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the least integral values (if any) of a, b, and c that will make $2(a+b+c)\pm 2\sqrt{[12ab-3(a+b-c)^2]}$ a square number for either sign of the radical.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This is always the case if $2(a+b+c)$ is the square, and $12ab=3(a+b-c)^2$ or $4ab=(a+b-c)^2$.

Let $a=(n+1)c$, $b=(n-1)c$.

Then $4ab=4(n^2-1)c^2$, $(a+b-c)^2=(2n-1)^2c^2$.

$\therefore 4(n^2-1)=(2n-1)^2=4n^2-4n+1$. $\therefore n=\frac{5}{4}$, $a=\frac{9}{4}c$, $b=\frac{1}{4}c$,

$\therefore 2(a+b+c)=7c$.

$7c=$a square if $c=7$, 7^3, 7^5, 7^7, etc.

$\therefore a=63$, $b=7$, $c=28$ are the least numbers, and $2(a+b+c)\pm 2\sqrt{[12ab-3(a+b-c)^2]}=(14)^2$.

115. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Required the least three square integral numbers the difference between the sum of every two of them and the third shall be a square number.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x^2y^2, y^2z^2, z^2 be the numbers.

Then $x^2+y^2-1=a^2$.....(1), $x^2-y^2+1=b^2$.....(2). $\therefore x^2=\frac{1}{2}(a^2+b^2)$.

Let $a=t+u$, $b=t-u$. $\therefore x^2=t^2+u^2$.

Let $t=n(p^2-q^2)$, $u=2npq$. $\therefore x^2=n^2(p^2+q^2)^2$.

From (1), $y^2=a^2-x^2+1$.

$\therefore y^2=4n^2yq(p^2-q^2)+1=(2mn-1)^2$, suppose.

$\therefore n=\dfrac{m}{m^2-pq(p^2-q^2)}$.

Let $z=m^2-pq(p^2-q^2)$. Then $xz=m(p^2+q^2)$, $yz=m^2+pq(p^2-q^2)$.

$\therefore (x^2+y^2-1)z^2=m^2(p^2+2pq-q^2)^2$,

$(x^2-y^2+1)z^2=m^2(p^2-2pq-q^2)^2$,

$(y^2-x^2+1)z^2=2m^4+2p^2q^2(p^2-q^2)^2-m^2(p^2-q^2)^2$.

Let $m=p^2-q^2$. $\therefore (y^2-x^2+1)z^2=(p^2-q^2)^2(p^4+q^4-4p^2q^2)$.

Let $p^4+q^4-4p^2q^2=(p^2-2rq^2)^2$. $\therefore p^2=\dfrac{4r^2-1}{r-1}\cdot\dfrac{q^2}{4}$.

When $r=13$, $p^2=\frac{225}{16}q^2$, $m=\frac{209}{16}q^2$.

$\therefore xz=\frac{209}{256}(241)q^4$, $yz=\frac{209}{256}(269)q^4$, $z=\frac{209}{256}(149)q^4$.

Deleting the common factor $\frac{209}{256}$, $xz=241q^4$, $yz=269q^4$, $z=149q^4$, $(x^2+y^2-1)z^2=(329)^2q^8$, $(x^2-y^2+1)z^2=(89)^2q^8$, $(y^2-x^2+1)z^2=(191)^2q^8$.

Let $q=1$, and the least numbers all different are 241, 260, 149.

By giving q different values an infinite number of solutions can be found.

AVERAGE AND PROBABILITY.

137. Proposed by G. H. HARVILL, Malakoff, Texas.

A, B, C, and D, playing whist, agree that the person who first cuts an ace shall have a stake of \$313. What is the value of each person's expectation before the play begins, each taking his turn at cutting in the order named as the game progresses?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., J. E. SANDERS, Hackney, Ohio, and J. SCHEFFER, A. M., Hagerstown, Md.

The chance that A cuts an ace$=\frac{1}{13}$, that he does not$=\frac{12}{13}$.

The chance that B cuts an ace$=\frac{1}{13}(\frac{12}{13})$, that he does not$=(\frac{12}{13})^2$.

The chance that C cuts an ace$=\frac{1}{13}(\frac{12}{13})^2$, that he does not$=(\frac{12}{13})^3$.

The chance that D cuts an ace$=\frac{1}{13}(\frac{12}{13})^3$, that he does not$=(\frac{12}{13})^4$; etc.

$\therefore A$'s chance$=\frac{1}{13}[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots\ldots]=\frac{2197}{4825}$.

B's chance$=\frac{1}{13}(\frac{12}{13})[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots\ldots]=\frac{2028}{4825}$.

C's chance$=\frac{1}{13}(\frac{12}{13})^2[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots\ldots]=\frac{1872}{4825}$.

D's chance$=\frac{1}{13}(\frac{12}{13})^3[1+(\frac{12}{13})^4+(\frac{12}{13})^8+(\frac{12}{13})^{12}+\ldots\ldots]=\frac{1728}{4825}$.

A's expectation$=\frac{2197}{4825}$ of \$313$=\$87\frac{6886}{4825}$;

B's expectation$=\frac{2028}{4825}$ of \$313$=\$81\frac{939}{4825}$;

C's expectation$=\frac{1872}{4825}$ of \$313$=\$74\frac{6886}{4825}$;

D's expectation$=\frac{1728}{4825}$ of \$313$=\$69\frac{939}{4825}$.

138. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of, (1) triangle, (2) quadrilateral, (3) pentagon, (4) hexagon, formed by taking, (1) three, (2) four, (3) five, (4) six random points on the circumference of a given circle radius a.

Solution by J. E. SANDERS, Hackney, Ohio, and the PROPOSER.

We will solve for hexagon first. Let $\angle AOM=\theta$, $\angle BOM=\phi$, $\angle COM=\psi$, $\angle DOM=\rho$, $\angle EOM=\delta$.

Then $AO=2a\sin\theta$, $BO=2a\sin\phi$, $CO=2a\sin\psi$, $DO=2a\sin\rho$, $EO=2a\sin\delta$, where a=radius of circle.

Area of hexagon$=u=2a^2[\sin\theta\sin\phi\sin(\theta-\phi)+\sin\phi\sin\psi\sin(\phi-\psi)+\sin\psi\sin\rho\sin(\psi-\rho)+\sin\rho\sin\delta\sin(\rho-\delta)]$.

$$\therefore \Delta = \frac{\int_0^{\pi}\int_0^{\theta}\int_0^{\phi}\int_0^{\psi}\int_0^{\rho} u\,d\theta\,d\phi\,d\psi\,d\rho\,d\delta}{\int_0^{\pi}\int_0^{\theta}\int_0^{\phi}\int_0^{\psi}\int_0^{\rho} d\theta\,d\phi\,d\psi\,d\rho\,d\delta}$$

$$= \frac{120}{\pi^5}\int_0^{\pi}\int_0^{\theta}\int_0^{\phi}\int_0^{\psi}\int_0^{\rho} u\,d\theta\,d\phi\,d\psi\,d\rho\,d\delta$$

$$= \frac{120a^2}{\pi^5}\int_0^{\pi}\int_0^{\theta}\int_0^{\phi}\int_0^{\psi} [2\rho\sin\theta\sin\phi\sin(\theta-\phi)$$

$$+2\rho\sin\phi\sin\psi\sin(\phi-\psi)+2\rho\sin\psi\sin\rho\sin(\psi-\rho)$$

$$+\sin^2\rho-\rho\sin\rho\cos\rho]d\theta\,d\phi\,d\psi\,d\rho \ldots\ldots(1),$$

$$= \frac{30a^2}{\pi^5}\int_0^{\pi}\int_0^{\theta}\int_0^{\phi} [4\psi^2\sin\theta\sin\psi\sin(\theta-\phi)+4\psi^2\sin\phi\sin\psi\sin(\phi-\psi)+3\psi$$

$$-3\sin\psi\cos\psi-2\psi^2\sin\psi\cos\psi]d\theta\,d\phi\,d\psi \ldots\ldots(2),$$

$$= \frac{20a^2}{\pi^5}\int_0^{\pi}\int_0^{\theta} [2\phi^3\sin\theta\sin\phi\sin(\theta-\phi)+3\phi^2-\phi^3\sin\phi\cos\phi-3\sin^2\phi]d\theta\,d\phi \ldots\ldots(3),$$

$$= \frac{5a^2}{2\pi^5}\int_0^{\pi} [10\theta^3-15\theta-2\theta^4\sin\theta\cos\theta+15\sin\theta\cos\theta]d\theta = \frac{15a^2}{2\pi^5}\left(1-\frac{3}{\pi^2}\right).$$

For the pentagon the points can be taken $\frac{1}{5}$ the number of ways for a hexagon. Writing θ for ϕ in the three last terms of (3), we get for the pentagon,

$$\Delta = \frac{4a^2}{\pi^4}\int_0^{\pi} [3\theta^2-\theta^3\sin\theta\cos\theta-3\sin^2\theta]d\theta = \frac{5a^2}{\pi^4}\left(1-\frac{3}{2\pi^2}\right)$$

For the quadrilateral, the points can be taken $\frac{1}{2}\frac{1}{0}$ the number of ways for a hexagon. Writing θ for ψ in the last three terms of (2), we get,

$$\Delta = \frac{3a^2}{2\pi^3}\int_0^{\pi} [3\theta-3\sin\theta\cos\theta-2\theta^2\sin\theta\cos\theta]d\theta = \frac{3a^2}{\pi}.$$

For the triangle the points can be taken $\frac{1}{6}\frac{1}{0}$ the number of ways for a hexagon. Writing θ for ρ in the last two terms of (1), we get

$$\Delta = \frac{2a^2}{\pi^2}\int_0^{\pi} [\sin^2\theta-\theta\sin\theta\cos\theta]d\theta = \frac{3a^2}{2\pi}.$$

These last three follow at once from the fact that for the pentagon, quadrilateral, and triangle, we have three, two, and one term in the expression for the area, while the power of π is for the pentagon π^4, for the quadrilateral π^3, for the triangle π^2, in the denominator.

139. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines. Golden, Col.

Four points are taken at random on the surface of a given sphere; find the average volume of the tetrahedron formed by the planes passing through the points taken three and three.

Remark by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

No. 139 is the same as No. 130 for which I have sent a solution previously.

140. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines. Golden, Col.

Obtain the average area of a triangle formed by a tangent to the four—cusped hypocycloid and the coördinate axes.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a=radius of fixed circle. Then portion of tangent intercepted by coordinate axes=a. Area of triangle=$\frac{1}{2}xy$, subject to the condition $x^2+y^2=a^2$.

$$\therefore \text{Average area}=\tfrac{1}{2}\int_0^a x\sqrt{(a^2-x^2)}\,dx \Big/ \int_0^a dx=\tfrac{1}{6}a^2.$$

141. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Upon a circular table, radius r, a *variable* square plate is thrown at random. What is the probability that the plate will lie wholly on the table?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the given circle, radius r; $ABCD$ the square, center E; $AB=2x$, $OE=z$, $\angle OAE=\theta$. Then $AE=x\sqrt{2}$.

If the center of the square, E, falls on a circle center O and radius $(r-x\sqrt{2})$, the square will be wholly on the table. If E falls on a circle, center O and radius z, the plate will lie wholly on the table. $z=\sqrt{[r^2+2x^2-2rx\sqrt{(2)}\cos\theta]}$. The limits of x are 0 and $\frac{1}{2}r\sqrt{a}$; of θ, 0 and $\frac{1}{4}\pi-\sin^{-1}(x/r)=\theta'$. Let p=chance. Since the whole number of ways E can fall on the circle is πr^2, we get,

$$p=\frac{\pi\int_0^{\frac{1}{2}r\sqrt{2}}[r-x\sqrt{2}]^2\,dx}{\pi r^2\int_0^{\frac{1}{2}r\sqrt{2}}dx}+\frac{\pi\int_0^{\frac{1}{2}r\sqrt{2}}\int_0^{\theta'}z^2\,dx\,d\theta}{\pi r^2\int_0^{\frac{1}{2}r\sqrt{2}}\int_0^{\theta'}dx\,d\theta}$$

$$=\frac{\sqrt{2}}{r^3}\int_0^{\frac{1}{2}r\sqrt{2}}[r-x\sqrt{2}]^2dx+\frac{[2+\sqrt{2}]}{r^3}\int_0^{\frac{1}{2}r\sqrt{2}}\int_0^{\theta}z^2dx\,d\theta$$

$$=\frac{1}{3}+\frac{[2+\sqrt{2}]}{r^3}\int_0^{\frac{1}{2}r\sqrt{2}}\{[r^2-2x^2][\tfrac{1}{4}\pi-\sin^{-1}\frac{x}{r}]-2x[\sqrt{(r^2-x^2)}-x]\}dx=1-\frac{\sqrt{2}}{9}.$$

MISCELLANEOUS.

136. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

(1) Solve (to five places) the equations, $\sin(x+\frac{1}{6}\pi)=10\sin x$, and $a\cos\phi\log\sin\phi=p$ where a is small and positive, and $\varphi=\alpha+\kappa$, where κ is very small and α is not very small; (2) If $a\theta=b\varphi$ where a is prime to b, and $\sin\theta=p$, $\sin\varphi=q$, how many values of q are there for each of p? (3) if $2x=\sin^{-1}x$, show there is only one positive value of x, and find it.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) $\sin(x+\frac{1}{6}\pi)=\frac{1}{2}\sqrt{3}\,\sin x+\frac{1}{2}\cos x=10\sin x.$

$\therefore(20-\sqrt{3})\sin x=\cos x$, or $\tan x=\dfrac{1}{20-\sqrt{3}}=\dfrac{20+\sqrt{3}}{397}=.05474.$ $\therefore x=3°8'.$

$\cos\varphi\log\sin\varphi=p/a$ or $\log(\sin\varphi)^{\cos\phi}=p/a.$

$\therefore\sin^2\varphi=e^{2p\sec\phi/a}.$

$$1-\cos^2\varphi=1+\frac{2p}{a}\sec\varphi+\frac{4p^2}{2!a^2}\sec^2\varphi+\frac{8p^3}{3!a^3}\sec^3\varphi+\ldots.$$

$$-a/2p=\sec^3\varphi+\frac{p}{a}\sec^4\varphi+\frac{2p^2}{3a^2}\sec^5\varphi+\frac{p^3}{3a^3}\sec^6\varphi+\ldots.$$

By reversion of series,

$$\sec\varphi=-\left(\frac{a}{2p}\right)^{\frac{1}{3}}-\frac{1}{3}\left(\frac{p}{4a}\right)^{\frac{1}{3}}-\frac{p}{18a}-\frac{p^2}{81a^2}\left(\frac{a}{2p}\right)^{\frac{1}{3}}-\frac{29p^2}{2430a^2}\left(\frac{p}{4a}\right)^{\frac{1}{3}}\ldots.$$

The value of $\sec\varphi$ to five terms. Since a is small, p is still smaller. When any multiple of $\frac{1}{2}\pi$ is substituted for φ, $\cos\varphi\log\sin\varphi$ is a maximum.

(2) $\sin\theta=\sin(m\pi-\pi)=\sin(2m\pi+\theta)=p,$

$\sin\varphi=\sin(m\pi-\varphi)=\sin(2m\pi+\varphi)=q.$

Since $a\theta=b\varphi$, but one value of q satisfies one value of p; for $a(n\pi\pm\theta)=b(n\pi\pm\varphi)$ only when $a=b$ and $a\theta=b(n\pi\pm\varphi)$ never.

$$(3)\ x=\sin 2x=2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\frac{512x^9}{9!}+\ldots\ldots$$

$$\therefore y=0 \text{ or } 1=\frac{8x^2}{3!}-\frac{32x^4}{5!}+\frac{128x^6}{7!}-\frac{512x^8}{9!}+\frac{2048x^{10}}{11!}-\ldots\ldots$$

By reversion of series, $x=.94775$, nearly.
$2x=108^\circ\ 36'$, nearly. $x=54^\circ\ 18'$, nearly.

137. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

The first transvectant of the binary cubic and its second transvectant is the *cubicovariant* of the binary cubic.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and W. G. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon, Ill.

Let $u=ax^3+3bx^2y+3cxy^2+dy^3$, be the binary cubic. Then the second transvectant is

$$\frac{d^2u}{dx^2}\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dxdy}\right)^2=0=(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2.$$

This is the Hessian of the cubic.
Let $v=(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2$.
The first transvectant of u, v is

$$\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}=0.$$

$$\therefore 3(ax^2+2bxy+cy^2)[ad-bc)x+2(bd-c^2)y]$$
$$-3(bx^2+2cxy+dy^2)[(ad-bc)y+2(ac-b^2)x]=0.$$

$$\therefore (a^2d-3abc+2b^3)x^3+3(abd-2ac^2+b^2c)x^2y$$
$$+3(2b^2d-bc^2-acd)xy^2+(3bcd-ad^2-2c^3)y^3=0.$$

This is the cubicovariant of the cubic u.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

170. Proposed by J. F. LAWRENCE, A. B., Breckenridge, Mo.

Suppose the market value of 5% bank stock to be $11\frac{1}{9}\%$ higher than 8% corporation bonds; I realize 8% on my investment, and my income from each is \$180; what did I invest in each?

171. Proposed by JOHN S. ROYER, Editor of The School Visitor, Columbus, Ohio.

A drawer made of inch boards is 8 inches wide, 6 inches deep, and slides horizontally. How far must it be drawn out to put into it a book 4 inches thick, 6 inches wide, and 9 inches long?

ALGEBRA.

183. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the condition that $x : y : z$ may be real, given that $\Sigma ax^2 = \Sigma axx = 0$, and $\Sigma aa^2 = 1$.

184. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

If m rows, viz., the h_1th, h_2th, ..., h_mth, be transferred so as to become the 1st, 2d, ..., mth, without altering the relative positions of the remaining rows, and that n columns, viz., the k_1th, k_2th, ..., k_nth, be similarly transformed, the determinant thus obtained is the same as the original or differs from it only in sign according as $h_1+h_2+.....+h_m-\frac{1}{2}m(m+1)+k_1+k_2+.....k_n-\frac{1}{2}n(n+1)$ is odd or even. [*Muir.*]

GEOMETRY.

205. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Having given any two systems of conjugate semi-diameters of an ellipsoid, the parallelopiped which has any three for continuous edges is equal to that which has the other three for continuous edges.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$ABCD$ is circumscribed by a circle center O, and it circumscribes a circle radius r. The perpendiculars from C on the sides are x, y, z, u. Show that $\frac{1}{2}AC.BD = r\Sigma x$.

CALCULUS.

170. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find the center-locus of conics having 4-point contact with a given conic at a given point. Show that the conic of minimum eccentricity is given by $e^4ab^2\varphi+4e^2-4=0$, where e is its eccentricity, and φ is the angle which the linear center-locus above makes with the normal to the curve at the point.

171. Proposed by J. E. SANDERS, Hackney, Ohio.

A thread passes spirally around a *rough* cylinder 10 feet high and 6 inches in diameter. How far will a pigeon fly in unwinding the thread if the distance between the coils is 4 inches, and the thread *unwound* is at all times *horizontal*?

MECHANICS.

161. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Four equal uniform smoothly jointed rods length a, and weight w, form a rhombus $ABCD$, A and C being in contact with two vertical walls b feet apart. An elastic string, natural length x, modulus λ, keeps the figure in position. The angle of friction at A and C is $\tan^{-1}p$. When the rhombus is just about to slip, find the angle A, and the angle between AB and the vertical.

162. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Show that the velocity, v, of a wave along the surface of a liquid whose depth is not less than the wave-length, λ, whose density is δ, and surface tension, T, is $v^2=\frac{g\lambda}{2\pi}+\frac{2\pi T}{\lambda\delta}$.

DIOPHANTINE ANALYSIS.

118. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the two least integral numbers such that their sum shall be a square, and the sum of their squares a biquadrate.

AVERAGE AND PROBABILITY.

147. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud; England.

In a bag are n balls, known to be black or white, either color, *a priori*, equally likely. I draw two, which turn out to be one white and one black. I replace them and draw two more. What is the chance both are black?

NOTES.

Professor Lon C. Walker has been elected Professor of Mathematics in the Colorado School of Mines, Golden, Colorado.

Dr. Saul Epsteen, of the University of Chicago, has very kindly consented to edit, for the coming year, the problems and solutions for the MONTHLY. So beginning with this issue all problems and solutions should be sent to him. The MONTHLY is to be congratulated that it has fallen into the hands of such able mathematicians as Drs. Dickson and Epsteen. We predict for it the most prosperous year in its history. B. F. F.

ERRATA.

On page 171, problem 105, for "$a^{2m}+b^{2m}$" read $a^{2^m}+b^{2^m}$.

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ON THE GROUPS OF THE FIGURES OF ELEMENTARY GEOMETRY.

By PROFESSOR G. A. MILLER.

Among the various elementary illustrations of the group concept those which relate to the movements of space are perhaps the most instructive. Even those students who have no trouble in approaching the subject by analytic methods generally take pleasure in observing the geometric interpretation of some of the most useful groups. In what follows we shall aim to determine the groups of the most common geometric figures rather than to make use of these figures for the sake of illustrating known groups. Only the most elementary notions about group theory are presupposed. Only groups of finite order are here considered.

The group of a figure is composed of all the movements of space which transform the figure into itself. In these movements space is regarded as rigid, that is, any two points are transformed into two points at the same distance from each other as the original points.

All the movements which transform a system of points into itself must also transform a certain point into itself. This elementary fact may be proved as follows: There is at least one minimum sphere which includes all these points. If there were two distinct minima spheres having this property, all the given points would be in the space common to the two spheres. This is impossible since this common space is in a smaller sphere; viz., in the one whose radius is equal to the radius of the circle which is composed of the points common to the two equal spheres. Every movement which transforms a system of points into itself must therefore transform the center of the minimum circumscribing

sphere into itself. Hence such movements may be composed of rotations around this center and ever concentric sphere must be transformed into itself.

We shall first consider the rotations which transform a plane triangle into itself. If the triangle is equilateral there are three such rotations around the axis perpendicular to its plane at the center; viz., the rotations through 0°, 120°, and 240°, respectively. These three rotations clearly form the group of order 3. This, however, is not the group of the triangle, for there are three other rotations which transform it into itself; viz., the rotations around the lines of symmetry through 180°. The group formed by these six rotations (the equilateral triangle group) is evidently simply isomorphic with the group formed by all the possible permutations of three things,—the symmetric group of degree 3. It may be observed that any two of these rotations of period two are equivalent to one of period three and that the rotations of period two are non commutative; i. e., the rotation of period three which is equivalent to two rotations of period two depends upon the order in which they are taken. Hence the equilateral triangle group is non-commutative. It will be found that all the groups of the regular plane polygons and of the regular solids are non-commutative. All the groups of the spherical polygons are commutative. In fact, they are always cyclic.

If a triangle is isosceles without being also regular, its group is clearly of order two; while the identity is the group of the scalene triangle. The cyclic group of order 3 is the group of the equilateral spherical triangle. All other spherical triangles evidently admit no rotation besides the identity. Hence we see that *the group of the plane equilateral triangle is of order 6, that of the spherical equilateral triangle is of order 3, that of the plane isosceles triangle is of order 2. The other triangles admit no rotations besides the identity.*

Before considering the groups of movements of the quadrangles it may be desirable to consider the subject from a somewhat more general standpoint. Either all or just half of the movements which transform into itself a system of co-planar points finite in number consist of rotations around an axis perpendicular to the plane through the center of the minimum enclosing circle. Among these rotations there must be a smallest one and all the others are composed of repetitions of this smallest rotation. Hence all such rotations form a cyclic group. If there is any additional rotation it must be of period two. If this rotation is followed by each of the rotations of the given cyclic subgroup, in order, we obtain just as many distinct rotations of order two as there are operations in this subgroup. A group which contains a cyclic subgroup of half its order and only operations of order two besides those in this subgroup is known as a dihedral rotation group. *All the groups of movements of systems of co-planar points, finite in number, must therefore be either cyclic or of the dihedral rotation type.* It is easy to see that the dihedral rotation group of order $2n$ is the group of the regular plane polygon of n sides. On this account the dihedral rotation groups may be called the regular polygon groups. If s is any operation in the cyclic subgroup of order n and if t is any other operation of the group it follows that $st=$ some t and hence $stst=1$. Hence $t^{-1}st=s^{-1}$; that is, t transforms every s into its